Lessons 5-1 & 5-2 Extreme Values & the MVT Key

AP Calculus AB
Lessons 5-1 & 5-2: *Extreme Values & the MVT*

Name ~~HCml~~ 2016
Date

Learning Goals:

- *I can find local or global extreme values of a function.*
- *I can apply the Mean Value Theorem and find the intervals on which a function is increasing and decreasing.*

While we have already covered the first and second derivative test to find extrema and concavity of a function, we did so without some formal definitions. Just to make sure we can keep up the conversation with our mathematician friends, let's go through those definitions:

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) absolute maximum value on D if and only if $f(x) \le f(c)$ for all x in D.

(b) absolute minimum value on D if and only if $f(x) \ge f(c)$ for all x in D.

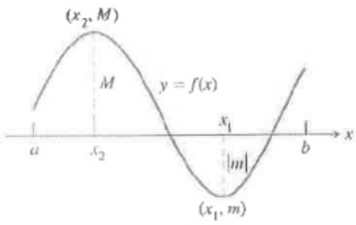
* *note that we usually leave off the term "absolute" or "global", so "minimum value" it is understood to mean absolute minimum – if referring to local minimums, local or relative needs to be specified.*

Explain the meaning of "$f(x) \le f(c)$ for all x in D."

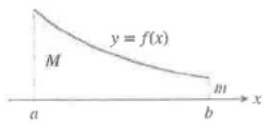
No matter what value is substituted for x, the value of f(x) will be less than f(c).

Extreme Value Theorem

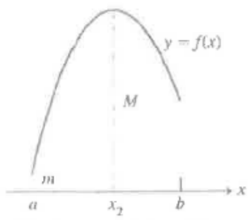
If f is continuous on closed interval $[a,b]$, then f has both a maximum value and a minimum value on the interval (see figure below).



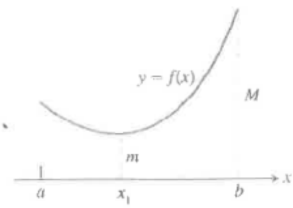
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

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DEFINITION Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) local maximum value at c if and only if $f(x) \le f(c)$ for all x in some open interval containing c.

(b) local minimum value at c if and only if $f(x) \ge f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Note that an absolute extrema is also a local extrema, because being an extreme value overall makes it an extreme value in its immediate neighborhood. Hence *a list of local extrema will automatically include absolute extrema*.

DEFINITION – Critical Point
A point in the interior of the domain of a function f at which $f'=0$ or f' does not exist is a **critical point** of f.

What do critical points tell you about f?

When f has a stationary point, relative max or relative min.

Practice

1. Find the absolute maximum and minimum values of $f(x)=x^{2/3}$ on the interval $[-2,3]$.

$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}$

$f'(x)\neq 0$ but is undefined when $x=0$.
Also check $x=-2$ and $x=3$ [endpoints]
$f(0)=0$ ← absolute min, → $(0,0)$
$f(-2)=(-2)^{\frac{2}{3}}=4^{\frac{1}{3}}$
$f(3)=(3)^{\frac{2}{3}}=9^{\frac{1}{3}}$ ← abs. max, → $(3,\sqrt[3]{9})$

2. Find the extreme values of $f(x)=\frac{1}{\sqrt{4-x^2}}$

$f(x)=(4-x^2)^{-1/2}$
$f'(x)=-\frac{1}{2}(4-x^2)^{-3/2}\cdot -2x$
$f'(x)=\frac{x}{(\sqrt{4-x^2})^3}$

Domain: $x\neq\pm2$
$f'(x)=0$ when $x=0$
$f(0)=\frac{1}{\sqrt{4}}=\frac{1}{2}$
Check endpoints: N/A Since interval is open
$\boxed{(0,\frac{1}{2}) \text{ absolute min}}$

3. Find the extreme values of $f(x)=\begin{cases}5-2x^2, & x\le 1\\ x+2, & x>1\end{cases}$

$\lim_{x\to1^-} f(x)=3$
$\lim_{x\to1^+} f(x)=3$ > cont.

Lhd: $-4x$
Rhd: 1 > not differentiable at $x=1$

Critical # at $x=1$
Critical # at $x=0$
$f(1)=3$ $(1,3)$ rel. min
$f(0)=5$ $(0,5)$ rel. max
Not abs. b/c the domain is open & end behavior is $-\infty$ & ∞

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THEOREM – Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b), then there is at least one point c in (a,b) at which

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

This theorem is another big one – *you will need to be able to refer to it by name.* Do not confuse it with the Intermediate Value Theorem!!

Speaking of the Intermediate Value Theorem, what does the IVT say?

Example

Show that the function $f(x) = x^2$ satisfies the hypothesis of the Mean Value Theorem on the interval $[0,2]$.

a b

Then find a solution to the equation $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

1. $f(x) = x^2$ is continuous over its entire domain.
2. $f'(x) = 2x$, which is continuous over its entire domain

$$2c = \frac{f(2)-f(0)}{2-0}$$

$$2c = \frac{4-0}{2}$$

$$2c = 2 \longrightarrow \boxed{c = 1}$$

Example

Why do the following functions fail to satisfy the conditions of the Mean Value Theorem?

(a) $f(x) = \sqrt{x^2} + 1$ on the interval $[-1,1]$

(b) $f(x) = \begin{cases} x^3+3, & x<1 \\ x^2+1, & x \ge 1 \end{cases}$ on the interval $[-2,3]$

a. $f(x) = \sqrt{x^2} + 1 = |x| + 1$ ☆ Not differentiable at $x = 0$ (corner)

b. $\lim_{x \to 1^-} f(x) = 1^3 + 3 = 4$

$\lim_{x \to 1^+} f(x) = 1^2 + 1 = 2$

$\neq$ ∴ not continuous

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A Physical Interpretation

If a car accelerating from zero takes 8 seconds to go 352 feet, its average velocity for the 8-second interval is 352/8 = 44 feet per second, or approximately 30mph. What does the Mean Value Theorem tell us about the speed of the car during that 8 seconds?

At some point during the 8 seconds, the car is travelling 30 mph.

Interesting historical note: New York State Throughway

In general, what does the Mean Value Theorem say about instantaneous change and average rate of change over some interval?

At some point during the interval, the average rate of change is equal to the instantaneous rate of change.

A Corollary of the Mean Value Theorem

Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval *I*, then there is a constant *C* such that $f(x) = g(x) + C$ for all *x* in *I*.

What does the above corollary mean?

If the derivatives of 2 functions are equal, the functions differ by a constant.

Let's apply the corollary

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

If $f(x) = F'(x) = \sin x$, then $F(x) = -\cos x + C$

$$2 = -\cos(0) + C$$
$$2 = -1 + C$$
$$C = 3$$
$$\boxed{F(x) = -\cos x + 3}$$

DEFINITION – Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all *x* in the domain of *f*. The process of finding an antiderivative is **antidifferentiation.**

Example

Find the velocity and position functions of an falling from an initial height of 20 feet $h(0)$ with an initial velocity $v(0)$ of −3 ft/sec. Remember, the acceleration due to gravity of a *falling* object is −32 feet/sec²

$a(t) = -32$

$v(t) = -32t + C$

$-3 = -32(0) + C$

$C = -3$

$v(t) = -32t - 3$

$h(t) = \frac{-32}{2}t^2 - 3t + C$

$20 = -16(0)^2 - 3(0) + C$

$C = 20$ $\boxed{h(t) = -16t^2 - 3t + 20}$

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Practice
No Calculator

1. A local minimum value of the function $y = \frac{e^x}{x}$ is

(A) $\frac{1}{e}$ (B) 1 (C) -1 (D) e (E) 0

$$y' = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

$x = 1$

e^x: + | +
x^2: + | +
$x-1$: − 0 +
− 1 + (min)

2. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when

(A) $y = 0$ (B) $y = \pm\sqrt{3}$ (C) $y = \frac{1}{2}$

(D) $y = \pm 3$ (E) none of these

$$2y \cdot \frac{dy}{dx} - y - x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - x) = y$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

← must be zero

$2y - x = 0$
$x = 2y$

$y^2 - 2y \cdot y + 9 = 0$
$y^2 - 2y^2 + 9 = 0$
$-y^2 = -9$
$y^2 = 9$
$y = \pm 3$

3. The function $f(x) = x^4 - 4x^2$ has

(A) one relative minimum and two relative maxima
(B) one relative minimum and one relative maxima
(C) two relative maxima and no relative minimum
(D) two relative minimum and no relative maxima
(E) two relative minimum and one relative maxima

$f'(x) = 4x^3 - 8x$
$0 = 4x(x^2 - 2)$
$x = 0 \quad x = \pm\sqrt{2}$

$4x$: − | − | + | +
$x+\sqrt{2}$: − | + | + | +
$x-\sqrt{2}$: − | − | − | +
− $-\sqrt{2}$ + 0 − $\sqrt{2}$ +
MIN MAX min

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4. Let $f(x) = x^3 - 3bx$ with $b > 0$.

(a) Find the coordinates of the relative maximum and minimum points of f in terms of b. Identify each specifically.
(b) Show that for all values of $b > 0$, the relative maximum and minimum points lie on a function of the form $y = -ax^3$ by finding the value of a.

a. $f'(x) = 3x^2 - 3b$
$0 = 3x^2 - 3b$
$\cancel{3}b = \cancel{3}x^2$
$\sqrt{b} = \sqrt{x^2}$
$x = \pm\sqrt{b}$
$3(x^2 - b)$: $+$ | 0 at $-\sqrt{b}$ max | $-$ | 0 at $\sqrt{b}$ min | $+$

Min:
$f(\sqrt{b}) = \sqrt{b}^3 - 3b\sqrt{b}$
$= b^{\frac{3}{2}} - 3b \cdot b^{\frac{1}{2}}$
$= b^{3/2} - 3b^{3/2}$
$= -2b^{3/2}$
$(\sqrt{b}, -2b^{3/2})$

Max:
$f(-\sqrt{b}) = -\sqrt{b}^3 - 3b(-\sqrt{b})$
$= -b^{3/2} + 3b \cdot b^{\frac{1}{2}}$
$= -b^{3/2} + 3b^{\frac{3}{2}}$
$= 2b^{3/2}$
$(-\sqrt{b}, 2b^{3/2})$

$-2b^{3/2} = a(\sqrt{b})^3$
$-2b^{3/2} = -ab^{3/2}$
$-2 = \bar{a}$
$\boxed{a = 2}$

$2b^{3/2} = -a(-\sqrt{b})^3$
$2b^{3/2} = ab^{3/2}$
$\boxed{a = 2}$

Calculator Active

5. The equation $y = \dfrac{895598}{1 + 71.57e^{-0.0516x}}$ predicts the population of Alaska since the year 1900 ($t = 0$ is 1900).

(a) Predict the population of Alaska in 2020.
(b) Find the inflection point of the equation. What significance does the inflection point have in terms population growth in Alaska?
(c) What does the equation indicate about the population of Alaska in the long run? Mathematically, why is this true?

a. $t = 2020 - 1900 = 120$
$y|_{120} \approx 781{,}253$

b. Graph $\dfrac{d^2y}{d^2x}$ & find the zeros.
The zero is $\approx t = 82.76$, so 1983
The P.O.I. is where the graph changes from CC↑ to CC↓. This point (1983) is when the population of Alaska was growing the fastest. After this point the growth rate started to increase at a decreasing rate.

c. $\lim\limits_{x \to \infty} \dfrac{895{,}598}{1 + 71.57e^{-.0516x}} = 895{,}598$
So the population of Alaska will level off at $\approx 895{,}000$.
(FYI ⇒ logistic growth model)